Improving mathematics learning in numbers and algebra (IMLNA) – a current project[1]

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Abstract: I will begin this reading with a brief presentation of the project (aims and organization). Afterwards I will present some results already reached in one of the three subgroups of the project (Task 1) concerning students' generalization. In this process, most of the students build a *mental* rule, verbalize it and write the rule in natural language, but do not get to symbolize the generalization. The students give sense to the letter as an unknown, but they have difficult in interpreting it as a generalized number. Finally, some questions are posed.

Key-words: algebraic thinking; letter; sequence; equation; generalization.

THE PROJECT IMLNA

The aims

In most countries, numbers and algebra are two fundamental topics of the school mathematics curriculum. Numbers have a decisive role in mathematics learning in early years and algebra is a key mathematical topic from the intermediate years onwards. This project aims to contribute towards a better understanding of the reasons that yield Portuguese students to have low achievement in these areas and to identify what can be done to improve their learning.

This project aims to contribute towards (i) a better understanding of the difficulties of students in learning numbers and algebra and of the potential of the teaching approaches based in innovative strategies; (ii) disseminating results concerning these aspects and promoting their discussion by the national and international research community; (iii) providing suggestions, recommendations and relevant examples to those responsible for writing the official mathematics curriculum, to textbook authors and those in charge of pre-service and in-service mathematics teacher education; and (iv) promoting the professional development of project members, and those who contact closely with it or have the opportunity of participating in teacher education and dissemination sessions.

The algebraic thinking

In this view, algebraic thinking includes:

- the ability to deal with algebraic computations and functions;

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- the ability to deal with mathematical structures and using them in the interpretation and solving of mathematical or extra mathematical problems;
- the manipulation of symbols, using them in creative ways in describing situations and in solving problems.

So, in algebraic thinking attention is given not only to objects but also to existing relations among them, representing and reasoning about those relations in a way as general and abstract as possible. Therefore, one of the most important ways to promote this reasoning is the study of patterns and regularities.

The learning and the teaching

This project presumes that students' learning trajectories may be strongly assisted by using appropriate teaching strategies based on exploratory and investigative work, using realistic learning situations, new information and communications technologies (ICT) and adequate representation systems. The teacher has here a new educational role – stimulating students' mathematical activity and synthesizing collective mathematical validations – along with the classical role of providing information and mathematical knowledge.

The organization

The project IMLNA involves four tasks (T1, T2, T3 and T4) – three to be developed in three small groups and a fourth involving all teams. Each of the first three aims to study the comprehension and the difficulties of students from different age groups (from the 5th to the 11th grade and at university level, 2nd year) in key numerical and algebraic topics (rational numbers, proportion, functions, patterns and relationships and numerical analysis). It also seeks to analyse the potential of an approach based on strategies of problem solving, exploration, investigation, use of ICT (thus facilitating working with different representations), and using real life situations. Besides the national and international literature, in each of these tasks empirical work will be carried out. Teaching units according to these strategies will be developed including tasks about rational numbers, proportion, patterns and relationships, numerical estimation, number sense, symbol sense, functions and numerical analysis, their representations and connections. Data gathering instruments concerning classroom observations, interviewing students and reflecting with teachers will be developed.

In the fourth task, involving the participation of two consultants (a Brazilian and an Italian), a global analysis will be carried out of learning trajectories in the field of numbers and algebra along the 2nd and 3rd cycles of basic education and secondary school, as well as a global evaluation of the teaching strategies applied. Working instruments will be prepared, both for the teaching experiments and for data collection and analysis and there will be discussions about each team's reports. Additionally two seminars will be held and open to outside researchers and teachers. Another important aspect of this task is the collective discussion of papers to publish in national and international scientific journals as well as a book about the teaching units covered in the project (including the tasks developed and their role in the curriculum). Another important concern of the project is sharing ideas and results with the national

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and international scientific community and promoting the professional development of both researchers and teachers involved in a direct or indirect way.

RESULTS ALREADY REACHED (T1)

The letters and the generalization

Pereira & Saraiva (2009) studied a class of grade 7 students. The problem of the study was to identify the learning and the difficulties that students present concerning the comprehension and the use of letters when they investigate involving generalization.

The state of the art

Research shows that many students have great difficulties in numbers and operations. Other students obtain here a reasonable level of performance, but later come across with great difficulties in algebra. One of the reasons of these difficulties is related to the diverse subtleties and changes of meaning of symbols when one moves from one field to the other (Usiskin, 1988). Another difficulty is related to the symbolic understanding of the numerical and algebraic expressions and their connections (Schoenfeld, 2005). A student trained to answer only to algorithmic questions is hardly able to deal with questions that aim at a conceptual understanding or that involve a combination of representations.

The difficulties of the students in the transition of arithmetic to algebra have been studied by numerous authors (for example, Booth, 1994; Rojano, 2002). They include (i) giving meaning to an algebraic expression; (ii) failing to see a letter as representing a number; (iii) attributing concrete meanings to letters; (iv) translating information from natural to algebraic language; (v) understanding the changes of meaning of the symbols + and = from arithmetic to algebra; and (vii) failing to distinguish arithmetic (3+5) from algebraic addition (x+3). Up to the present, school mathematics has emphasized the teaching of algorithms and computation procedures. However, teaching these algorithms when the students do not yet grasp the meaning of the operations leads to a mechanization without understanding that yields to weak performances as well as to an attitude of rejection of mathematics (Rojano, 2002). Important curriculum initiatives acknowledge these problems. For example, the influential NCTM (2000) document regards both numbers and algebra as a basic part of the school mathematics curriculum.

Algebra involves strong symbolization. In fact, symbolization begins in arithmetic. In recent years, symbolism has been downplayed. However, symbolism is an essential part of mathematics that cannot be excluded. In fact, on the one hand, symbols have great value since they agglutinate ideas in compact aggregates, transforming them in information easy to understand and manipulate (Sfard & Linchevski, 1994). On the other hand, symbolism leads easily to formalism when we lose of sight the meanings that the symbols represent and only give attention how to manipulate them (Davis & Hersh, 1995), thus hampering the learning process. It is necessary, therefore, to find a

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road in teaching and learning numbers and algebra that provides an accessible and productive entrance both to mathematical language and to mathematical understanding.

Usiskin (1988) says that the school algebra has to do with the understanding of "letters" (today we usually call them *variables)* and their operations, and considers the five following equations to illustrate the different meanings to the letters:

1. $A = LW$
2. $40 = 5x$
3. $\sin x = \cos x \bullet \tan x$
4. $1 = n \bullet (l/n)$
5. $y = kx$

To him, each of these equations has a different feel. We usually call (1) a formula, (2) an equation (or open sentence) to solve, (3) an identity, (4) a property, and (5) an equation of a function of direct variation (not to be solved). To Usiskin, these different names reflect different uses to which the idea of variable is put. In *(1), A, L,* and *W* stand for the quantities area, length, and width and have the feel of knowns. In (2), we tend to think of *x* as unknown. In (3), *x* is an argument of a function. Equation (4), unlike the others, generalizes an arithmetic pattern, and *n* identifies an instance of the pattern. In (5), *x* is again an argument of a function, *y* the value, and *k* a constant (or parameter, depending on how it is used). Only with (5) is there the feel of "variability," from which the term *variable* arose.

To Mason, Graham & Wilder (2005), the algebraic thinking, particularly the recognition and articulation of generality, is within reach of all learners, and vital if they are to participate fully in society (p. ix). Yet, every learner who starts school has already displayed the power to generalize and abstract from particular cases, and this is the root of algebra. To those authors, the expressing generality is entirely natural, pleasurable, and part of human sense-making. Algebra provides a manipulative symbol system and language for expressing and manipulating that generality (p. 2). However, many students have great difficulties in algebra, particularly in problem solving involving symbolic generalizations. The meaning of the symbols students make is frequently without sense – it is only a memory process.

Many authors, such as Rojano (2002), say that the generalization process has a first moment of perception of generality, which consists, for example, in the recognizing of a pattern in a numeric sequence – it is a mental process, and it happens, for instance, when the students are able to get any term of a sequence without the necessity to extend the terms of the sequence to that order; a second moment of the expression of generality, elucidating a general rule, verbal or numeric, to generate a sequence – it is a mental rule presented in natural language, or numerically; a third moment which is the symbolic expression of generality, yielding a formula corresponding to the general rule; and a forth one of the manipulation of the generality, solving problems related to the sequence. This is a cycle that must be seen in a flexible way, but it contains the essential moments in the generalization process.

As Ponte (2006) indicates, the analysis of the mathematics curriculum of Portugal and other countries, in numbers and algebra, raises questions regarding its intuitions and basic models, main structural concepts, basic representations, study of algorithms and role of technology. Kaput & Blanton (2005) suggest that is necessary to experiment curricula that combines (i) promoting representation and thinking processes that seek generalization whenever possible; (ii) treating numbers and operations algebraically, giving attention to existing relations (and not just to the numerical values) as formal objects for algebraic thinking; and (iii) promoting the study of patterns and regularities, from as early as possible. On the other hand, the algebraic structure and symbolism can be building from the mathematical experience with numbers, emphasizing the intuitive and the strategic aspects (NCTM, 2000; Guzmán, 1996). Also, an investigative approach, including the visualization and the manipulation of figures, is considered a good support to the generalization, because it can allow the students to the building of an algebraic formula (Kieran, 2006).

Pedagogical proposal

The pedagogical proposal was elaborated with tasks that promote the generalization, the resolution of equations, and problem solving including equations. Investigative, exploratory, problems and exercises (these one from the text-book) tasks were proposed to the students. Some tasks were formulated in pure mathematical terms and others were semi-real (in the sense of Skovsmose, 2000) – real at first sight but, in practice, conditioned to a didactic contract established with the students about the acceptance of the conditions of the statement of the tasks that are relevant to the resolutions; so, there are many real characteristics of the objects referred in the statement that are not considered as in a pure real task.
By this way, the first task, an investigative one, *John's birthday*, is put in the *Sequences* theme. The second one, *Discovering the value of the letters*, an exploratory task, is put in the *Equations* theme. The third task, *The test of evaluation* (*TE*), with the investigative task *The tower of the odd numbers*, aims to think over the students' learning, and it is put in the *Sequences* theme. The fourth task, a problem, *The money-pots*, is put in the *Problem solving including equations* theme.

Methodology

In this study a qualitative and interpretative approach was followed (Bogdan e Biklen, 1991). The teacher, Magda Pereira, simultaneously assumed the role of teacher and investigator. The empirical work was realized during the second period of the school year of 2007/2008, with a class, grade 7, fifteen students, during 11 classes with 90' each one. The students worked in group (three of them with four and one group with three students), in the mathematical *Numbers and Calculus* theme.

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The data collected: i) the students' resolutions of the sequenced tasks proposed by the teacher; ii) the dialogues in the class, between teacher and students, recorded by the teacher; iii) the students' resolutions of the test of evaluation (TE), and iv) the interviews made in group (two groups – GA and GB – with three students each one) in the end of the teaching of the *Numbers and Calculus* theme. To GA was proposed the exploratory task *The three twins*, and *The problem of the ages*; to GB was proposed the exploratory task *The messages of mobile*, and *The problem of the three brothers*.

The data analysis: They were considered two categories of analysis – *the ability to generalize*; and *the meaning of the letters*.

Results

<u>*The ability to generalize*</u>

In the beginning of the study, the students represent their reasoning by schemes, establishing relations between the data. In the first task (*Jonh's birthday:*

> *John organizes at his home a party in the day of his birthday. We don't know how many friends go to the party. However, we know that John will be at the door of his home to receive his friends - while they will be bringing near, they will greet John with a handshake, as well as each one of the friends who have already arrived. Only when they all are joined they enter at home.*
> *How many handshakes will be there before John and his friends enter at home?*),

the students started to handshake one each other, and recording what happened when the number of friends was increased (figure 1)

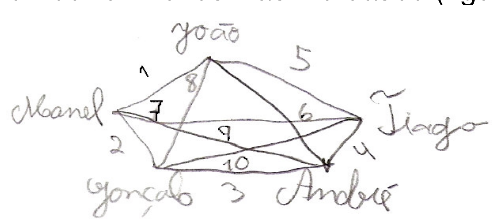

Fig. 1: The scheme *Jonh' s birthday (Class, Group A)*

By the scheme, the students identify and record the number of the handshakes to a specific number of friends, but they are not able to generalize it, even in the natural language. Also the students are not able to build any symbolic expression modeling the situation.

The teacher, in the whole class discussion, suggested the following table (figure 2):

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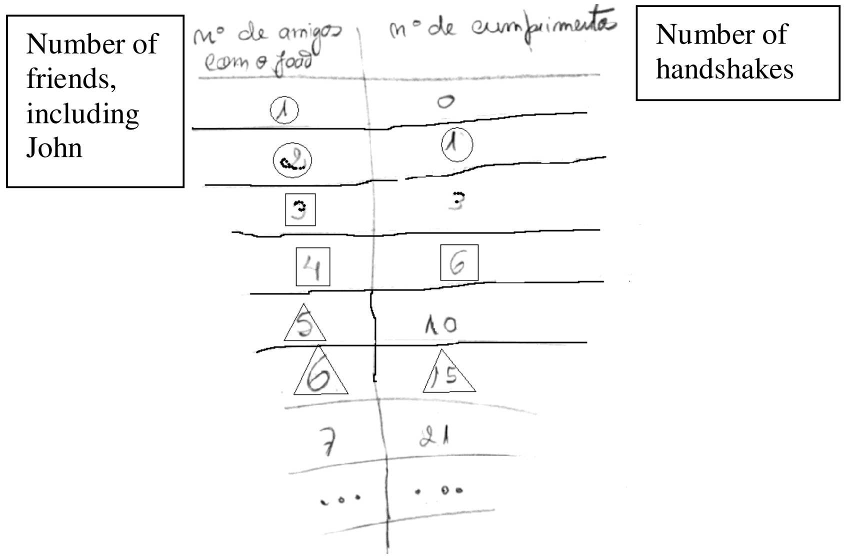

Fig. 2 – Table with values written by the teacher on the blackboard (2008/01/16)

The whole class discussion:

Teacher: What happens inside the circles? We can obtain 1, on the right, because we have, on the left…
Marco: I know! But, I don't know! I thought in repeated values, but that is only for the first situation. The values inside the squares are not as I was thinking.
Manuel: I have already figured it out. I figured out one thing, I think!
Teacher: Please say, Manuel.
Manuel: 3×4 are 12 and 12 ⁄ 2 are 6. It happens the same to the values that are inside the triangles. And it is the same to the others.
Teacher: Very well. And with a very big number of friends? How can we think?
Manuel: By the same way. For example, if we have 1000 friends, to know the number of handshakes … is, and then we will divide by 2.
Teacher: And if we will have an any number of friends?
António: So, it is that number times the number before it, and after we will divide by 2.

(Classroom, 16/01/2008)

With the teacher's help, the students separate from the small numbers of their initial schemes to the number 1000. They are able, even, to say with their own words how to calculate the number of handshakes. To the teacher's question about the eventual big number of friends, as big as we wish, Antonio says that the process is the same. However, the students are not able to get a symbolic expression of generality.
In this study, the students make evident their difficulties to get a symbolic expression of generality, yielding a formula corresponding to the general rule of the situation. In the lesson number 6 of this theme, students' evidence difficulties to write a symbolic expression of generality to the task *The tower of the odd numbers* (*Consider the table of numbers*

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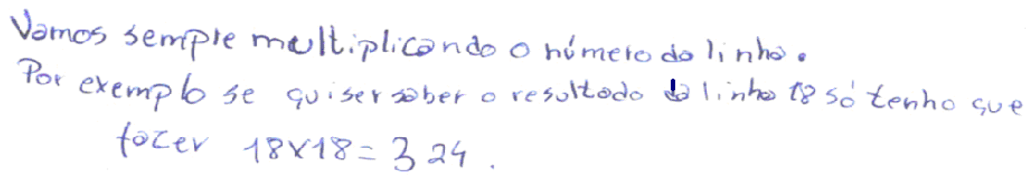
1
1 3
1 3 5
1 3 5 7
1 3 5 7 9
1 3 5 7 9 11
...

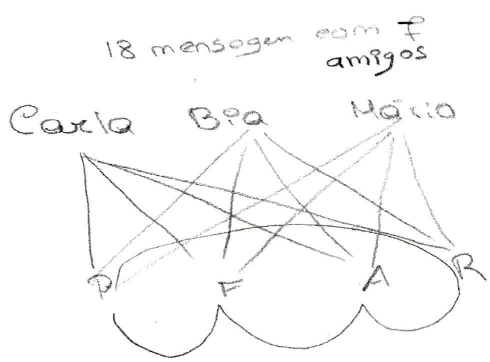
Say the value of the sum of the numbers of a line of this triangle according to the number of the line (descending order)). For instance, we present Rosas' answer (figure 3):



We go always multiplying the number of the line. For example, if I want to know the result about the line number 18, I must do 18×18=324.

Fig. 3: Rosa's answer

Rosa just gives a rule in her natural language, giving an example to the case 18. She doesn't present a symbolic expression of generality.
Even in the end of the teaching of the *Numbers and Calculus* theme, the students evidence some resistance to get a formula corresponding to the general rule of the situation. Most of them are satisfied with a schema, as we can see in the answer given by Group B, in the interview, to the question *The messages of mobile* (*Carla, Mário and Bia met, today, in a bakery. They drink tea and, in the end of their meeting, they decide, when they arrive to their homes, to send messages, by mobile, to invite some more friends to go with them to the cinema. They invited some friends, but we don't know how many of them will go to the cinema. We know that Carla, Mário and Bia send, each one, one message to the same group of friends. We also know that these friends exchanges one message between them. How many messages are sended?).*



18 messages with 7 friends

Fig. 4: The scheme presented by Group B (interview)

The students, by themselves, don't sense the generality. They consider essentially specific cases.

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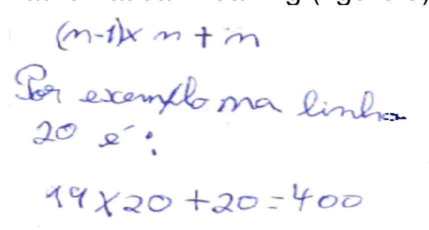
An intervention of the teacher is necessary to the students give the step forwards the symbolic expression of generality.

The meaning of the letters

In the resolution of first task (*John's birthday*), and after the students arrived to the number 1000, the use of the symbols only appears with the teacher's help, as we can see in the next report (Whole class discussion):

Teacher: And if we have an any number of friends?
António: So, it is any number times the number before it, and after we will divide by 2.
Teacher: So, and in a simple way, if we say that that number is *n,* how can we will find the number of handshakes?
Manuel: The number of handshakes is any other one, like *y*, for example.
Teacher: Well, that way, we don't know anything. I say that *n* friends went to the party, and you say that there were *y* handshakes. By this way, do we get any information?
Nelson: And with *n* we also don't know anything, because we need to know how many friends are *n*.
Teacher: How did we think of 1000 friends? Can we think the same way for *n* friends?
António: Even without knowing how much is *n*?
Rosa: I think I know! It is *n* times the number before it, that is…
António: So, can we multiply numbers without knowing their values? And, afterwards, how do we know the result?
Rosa: The result depends of how many friends are *n.*
(Classroom, 16/01/2008)

The students' difficulty is concerns the value of *n* – they don't know it. How can they reason with *n* without knowing its value?
In the task *The tower of the odd numbers*, the most part of the students explain by own words what happened in a certain line using the natural language. However some of them solved the task appointing the number of any line of the tower through a letter, seeming that they had built a symbolic relation with mathematical meaning (figure 5):



For example, in line 20 is:

Fig. 5: Manuel's answer(TE)

Manuel needed to present a specific case to confirm his symbolic expression. Besides, the process he uses, as well as the symbolic expression, is similar to the *John's birthday* one, revealing a transposition of the processes used

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previously – which emphasizes the usefulness of the own mathematical experience.

Because in a generalized expression the letter represents a number – it can assume various values, and for that very reason the same to the result of the expression – is an obstacle to the students. They have an inclination to consider the letter as a static and unique unknown value, exactly the same as the letter in an equation. The next small report of the interview (GA) – discussing again the task *The messages of mobile* – is an example of this:

Teacher: So, using the letter *n*, what is the result of that reasoning?
António: I would like to find a way to build an equation, but I don't know what the unknown is. I think that the unknown is the number we go always increasing to the number of messages, as the number of friends increases, but I don't know.
Teacher: Please think a little more. You have a table. What does happen as the number of friends increases?
Marco: If we have *n* friends, …, no, …, *n* messages. I don't know what we are looking for.

(...)

Mara: I think I have one way. It is similar to a task we solved in class, some weeks ago. For example: $5\times 4 = 20$. Afterwards $20 \div 2$ are 10. But we want 7, so we must subtract 3.
Teacher: So, and to an any number of friends?
António: Ok, that is what I want, but I don't know which the unknown is, because the number of friends increases, and also the number of messages, but they don't increase in the same way. What is the unknown?

(...)

Group A (interview)

The students make evident a strong resistance to the building of a generalization. The difficulty seems to be in the interpretation that the students make of a letter in the various situations. Along the study the students promote more familiarity with the letter as an unknown – as an equation, the letter is the unknown value that we must calculate – than as a generalized number.

Conclusions

The students make sense of the letter when it assumes the role of unknown. When the letter is inserted in a functional context they manifest lots of difficulties, especially to write a symbolic expression to generalize the general term of a sequence. However, the students are able to calculate the first terms of a sequence, and they are able to verbalize, and to write, in their natural language the general rule of the generalization.
In the process of generalization, most of the students build a *mental rule*, verbalize it and write the rule in natural language – but they do not get to symbolize the generalization. The students stay in the second stage of the generalization referred by Rojano (2002). Perhaps on account the ambiguity of

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the symbols. In fact, *n,* individually, not being a natural number represents all of them (Caraça, 1998).

The schemes and the tables inserted in the resolution of tasks with an exploratory and investigative nature, and with a discussion students/students and students/teacher, promote the calculation of the first terms of a sequence, as well as the description of the general rule of the generalization using the preceding term. However, they did not promote the writing of the symbolic generalized expression of a sequence.

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FINAL NOTES

In this point, we pose some questions that are a challenge for our future research:

What is the role played by the schemes and by the visualization on the reasoning for generalization? What kind of table is more useful?

Why do students believe more in the meaning of the letter as an unknown than as a generalized number?

In what sense are the different meanings of the sign "=" related with the difficulties that the students evidence to interpret the letter as a generalized number in a symbolic expression context?